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The method of ascending symmetry for irreducible characters of finite groups

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The irreducible characters of a finite group are determined uniquely by those of a minimal set of maximal subgroups. The method is based on the construction of all class functions which are irreducible characters on every maximal subgroup. These are generalized characters by a theorem of Brauer, so that the irreducible characters are obtained by checking the norm. An alternative characterization of irreducible characters, the Maximum Mixing Rule, works for all point symmetry groups, and its physical significance is discussed. As an example, the character tables for all point symmetry groups and crystal double-groups are constructed in this way.

Key words: Character tables — Finite groups — Generalized and irreducible characters — Symmetry — Maximal subgroups — Mixing

1. Introduction

Character tables are a major tool in the application of group theory to physics [1–8] and chemistry [9–11]. Their construction is based on various results of group representation theory [12–15]. For example, a character table can be built [16] using the knowledge of the structure coefficients of the class algebra, with the help of the orthogonality theorems, by reduction of characters obtained by different ways (e.g. induction), and other methods.

The "Method of Descending Symmetry" is quite popular in the physical literature [4, 9, 10]. It describes the splitting of levels due to a symmetry-reducing distortion,

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by reduction of the representation subduced on a subgroup. Here we consider the inverse process of "Ascending Symmetry". We show that a character table of a finite group G is uniquely determined from the character tables of its (maximal) subgroups. This is done on two levels. On a rigorous mathematical level (Sect. 2) the result follows from a theorem by Brauer. On a more heuristic level, we suggest (Sect. 3) a "Maximum Mixing Rule" for determining the irreducible characters of G. It shows that character tables are constructed in a way that ensures a maximal removal of degeneracy when descending in symmetry.

The above mentioned results lead to a new method for constructing character tables, which we call the "Ascending Symmetry" or "Composition" method. It is demonstrated for all point symmetry groups (Sect. 4) and crystal double-groups (Sect. 5). Group theoretic notions used in the sequel are defined in Appendix A.

2. Theory

In the following presentation we restrict attention to nonabelian groups. This is due to the fact [17] that any abelian group is a direct product of cyclic groups (whose orders are powers of primes). Its character table is obtained by a direct multiplication of the tables for its cyclic components. The irreducible representations of a cyclic group of order n are the n powers of $(\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^{n-1})$ where $\varepsilon \equiv \exp(2\pi i/n)$ is the nth root of unity.

Our aim is to show, loosely speaking, that if the character tables for (proper) maximal subgroups M_i of a finite (nonabelian) group G are given together with the fusion of the conjugacy classes of the M_i into those of G (i.e., one is told which classes of M_i are contained in a given class of G), it is possible to determine uniquely the character table of G. First we introduce a

Definition (*E-cover*)

An elementary cover of a finite group G is a set of subgroups $\{H_i\}_{i=1}^{m}$ so that

(a) Each conjugacy class of G has a representative in one of the H_i's.

(b) Each conjugacy class of elementary subgroups $E \subseteq G$ has a representative contained in one of the H_i's.

Note that when G is elementary, $\{G\}$ is an E-cover (and every E-cover contains G). When G is non-elementary, it always has a proper E-cover (i.e. all H_i's are proper subgroups), for example the set of all elementary subgroups (every element in G belongs to an elementary subgroup, e.g. the cyclic subgroup generated by it). In this case, the E-cover does not necessarily cover G, but the set of all conjugates of the H_i's does.

The above definition is inspired by a theorem [13-15], which is seldom mentioned in the physical literature [1-11].

Theorem (Brauer). *A class function of a finite group G is a generalized character if and only if it is a generalized character when subdued (restricted) to every elementary subgroup $E \subseteq G$.*

With the help of the above we can prove the following

Corollary. The composition method for determining irreducible characters.

The character table of a finite non-elementary group G is uniquely determined by (a) The character tables of the (proper) subgroups H_i in an E-cover; (b) The rule for fusing the classes of H_i into those of G.

Proof. Let us denote the set of all irreducible characters of G by $\mathrm{Irr}(G)$. If $\chi \in \mathrm{Irr}(G)$, we denote by ν the dimension (degree) of χ, i.e. $\nu = \chi(e)$, where $e \in G$ is the identity element. Our proof is also a description of the procedure for obtaining all $\chi \in \mathrm{Irr}(G)$.

First we note that if two of the subgroups are conjugate, they are isomorphic and have the same character table and fusion rules. We therefore assume, without loss of generality, that all H_i's are representatives of distinct conjugacy classes of subgroups.

For a given dimension ν we construct all class functions $\{\phi\}$ of G so that

(a) $\phi(e) = \nu$, and

(b) The restriction of ϕ to every H_i is a (possibly reducible) character of H_i.

This can be done in the following manner: For H_1 take all $\mathbb{N}$-linear combinations $\sum_k a_{1k}\psi_{1k}$ (i.e., a_{1k} are positive integers), of $\psi_{1k} \in \mathrm{Irr}(H_1)$, so that rule (i) is obeyed:

Rule (*i*). If $h_1, h_2, \ldots \in H_i$ are representative of different conjugacy classes of H_i, but belong to the same class of G, then

$$\sum_k a_{ik}\psi_{ik}(h_1) = \sum_k a_{ik}\psi_{ik}(h_2) = \cdots \tag{1}$$

For H_2 and any of the above-constructed $\mathbb{N}$-linear combinations, take all $\mathbb{N}$-linear combinations $\sum_k a_{2k}\psi_{2k}$, $\psi_{2k} \in \mathrm{Irr}(H_2)$, so that both Rules (i) and (ii) are obeyed:

Rule (*ii*). If $h_i \in H_i$ and $h_j \in H_j$ are conjugate in G, then

$$\sum_k a_{ik}\psi_{ik}(h_i) = \sum_k a_{jk}\psi_{jk}(h_j) \tag{2}$$

In particular, $\sum_k a_{ik}\psi_{ik}(e) = \nu$ for all i.

Continuing in this fashion for $i = 1, \ldots, m$ we generate all class functions ϕ (the two rules ensure that ϕ is indeed a class function) that are (by construction) characters for all H_i. We now observe that:

(a) Since a representation of a group G is also a representation when restricted to any subgroup of G, it follows that every $\chi \in \mathrm{Irr}(G)$, $\chi(e) = \nu$, is one of the ϕ's.

(b) By Brauer's theorem all the ϕ's are generalized characters. We can therefore apply the criterion for irreducibility

$$\sum_{g \in G} |\phi(g)|^2 \geq |G| \tag{3}$$

where $|G|$ denotes the order of G, and equality holds if and only if $\phi \in \mathrm{Irr}(G)$. Note that if $\phi(e) = 1$, $|\phi(g)|^2 = 1$ for all $g \in G$ by construction, therefore ϕ is guaranteed to be an irreducible character.

The procedure described above ensures that we have obtained all irreducible characters of G with dimension ν. If we start the procedure with $\nu = 1$, increasing ν by one at each stage, and terminate when the number of irreducible characters obtained equals the number of conjugacy classes, we would have a unique determination of the character table for G. Q.E.D.

The only case not covered by the above corollary is that of a nonabelian p-group (an example are dihedral groups of order 2^k, $k \geq 3$, see Sect. 4). We would like to determine its character table by the composition method using only its proper E-cover. For the one-dimensional (linear) class functions Brauer's theorem is clearly wrong if only proper E-groups are considered. In this case composition may give class functions which are not generalized characters. These could not be filtered out by Eq. (3) alone (but we could try orthogonality with the trivial representation, consistency with the group's multiplication table, etc.). For class functions of dimension $\nu \geq 2$ Brauer's conclusion is still true for the few cases we have checked.

Although the composition procedure works for any E-cover, it is natural to ask when would it be "most efficient". On the one hand we would like the E-cover to contain as many (conjugacy classes of) subgroups as possible: Additional subgroups may add restrictions *via* Rules (i) and (ii) and decrease the number of class functions ϕ to be considered. On the other hand, we prefer to deal with as few character tables as possible. These (seemingly contradictory) demands motivate the following

Definition (*ME-cover*)

A *ME*-cover is a minimal elementary cover of maximal subgroups, $M_i \subset G$.

It is clear that a *ME*-cover contains at most one representative from each conjugacy class of maximal subgroups. It need not contain all such representatives: If every elementary subgroup $E \subset M_j$ is already contained in one of the M_i, $i \neq j$, there would be a *ME*-cover which does not include M_j. In applications (Sect. 4) we always use the character tables of a *ME*-cover.

Additional rules

We list some additional "rules" to the two rules in the corollary. Their use is either optional or restricted to special cases. In the following we assume that $H \subset G$ is in the E-cover under consideration.

Rule (*iii*). Every $\psi \in \mathrm{Irr}(H)$ appears as a constituent of some $\chi \in \mathrm{Irr}(G)$. This could be proven either from Frobenius' reciprocity relation [15] or by restriction of the regular character of G. This rule may help in eliminating some of the class functions towards the end of our procedure.

Rule (*iv*). A class function ϕ determined at the νth stage of our algorithm, should not be a $\mathbb{N}$-linear combination of $\chi \in \mathrm{Irr}(G)$ obtained at the previous $\nu - 1$ stages. This rule can be used instead of Eq. (3) to eliminate the reducible characters. (We would still need to use Eq. (3) for the "truly generalized" characters, which have at least one negative integer coefficient a_k in the combination $\sum_k a_k\chi_k$, $\chi_k \in \mathrm{Irr}(G)$.)

Rule (v). If $H \lhd G$ (H is a normal subgroup), then by Clifford's theorem [13–15] the $\psi_k \in \mathrm{Irr}(H)$ with coefficients $a_k \neq 0$ (cf. Eqs. (1) and (2)), are conjugates in G. They all have the same dimension $\psi_k(e)$, which must therefore be a divisor of ν.

Rule (vi). If the subclass algebra with respect to H_i is commutative, G is simply reducible with respect to H_i, i.e. all the a_{ik}'s are zero or unity [18].

Rule (vii). If we seek only the faithful irreducible characters of G, then

(a) We can start the procedure from $\nu = 2$: All one-dimensional (linear) characters for $|G| \geq 3$ are non-faithful.

(b) For each subgroup H_i, the intersection of the kernels of all ψ_{ik} (whose $a_{ik} \neq 0$; cf. Eqs. (1) and (2)) must equal the identity.

We mention this last "rule" since it is possible to obtain the non-faithful $\chi \in \mathrm{Irr}(G)$ from the irreducible characters of the factor groups G/N, where N is a minimal normal subgroup of G. For completeness, we review this method in Appendix B.

3. The maximum mixing rule

We have used so far mainly condition (3) to "filter out" the irreducible characters. It is possible to suggest another "characterization of irreducible characters" that we call the Maximum Mixing Rule (MMR). It holds for all point symmetry groups and some other finite groups. At present we do not know whether it is valid for arbitrary finite groups. We mention the MMR because it seems to have an interesting physical significance.

Let $\{\phi\}$ be the set of all ν-dimensional generalized characters of a group G, ($\nu > 1$ if G is elementary) that are characters on every (proper) subgroup H in an E-cover (see Sect. 2). When subdued (restricted) to H (denoted ϕ_H) one has $\phi_H = \sum_k a_k\psi_k$, $\psi_k \in \mathrm{Irr}(H)$, a_k integer. Define a class functional on $\boldsymbol{a} = (a_1, a_2, \ldots)$ by

$$f_H(\boldsymbol{a}) = \sum_k a_k^2 \tag{4}$$

then the MMR states that only the ϕ's for which $f_H(\boldsymbol{a})$ is minimal will be irreducible characters.

The function f is a measure of the "mixing" of the "states" $(1, 2, \ldots)$. Hence it is an "entropy function" (it is actually a Rényi entropy of order 2) [19]. For example, if $\phi_H = \psi_1 + \psi_2$ and $\phi'_H = 2\psi_1$ then ϕ_H is more mixed, therefore $\phi \in \mathrm{Irr}(G)$ while $\phi' \notin \mathrm{Irr}(G)$.

The physical significance of subduction on a subgroup is that it describes the reduction in symmetry caused by some perturbation [4]. Excluding "accidental" degeneracies, all eigenfunctions belonging to a given quantum level form the basis for an irreducible representation of the system's symmetry group. Hence, loosely speaking, the MMR states that upon applying a perturbation, these degenerate states split to the largest extent possible. This also means that the total entropy of the system never decreases after applying a perturbation, which therefore gives rise to an irreversible process [20]. Consistent with all the irreducible representations of its subgroups, the irreducible representations of a physical group are "chosen" in a way that ensures the irreversibility of all natural processes.

4. Irreducible characters of point symmetry groups

A point symmetry group can be defined as a finite group which has a real faithful representation of dimension $\nu \leq 3$ (they are all subgroups of the three-dimensional orthogonal matrice groups). Physics textbook [1–11] usually classify them by their symmetry elements. Hence one has groups of types C_n, S_n, D_n, C_{nh}, C_{nv}, D_{nh}, D_{nd}, T, T_h, T_d, O, O_h, I and I_h. (This is Schoenflies' notation [10]. Also given in Table 1 is the notation of some mathematical textbooks [17].) Since many of these are isomorphic or direct products, it is helpful to classify them by order, listing together all isomorphic groups [21]. This is done in Table 1. All abelian groups are either cyclic (whose order is a power of a prime), or a direct product of such groups with a cyclic group of order 2. Some of the nonabelian groups are also obtained as direct products with C_2. For all nonabelian groups we list the conjugacy classes in a way that makes the isomorphism evident. Group elements are denoted by lower case symbols according to the symmetry operation they represent: $e-$ identity; $c_n - n$ fold rotation; $\sigma-$reflection; $s_n - n$ fold rotation-reflection; $i-$inversion. A notation [10] such as $8c_3$ means that the class contains eight c_3 elements. Let us recall that the size of a conjugacy class is the index of its centralizer. For example, the centralizer of a c_3 element in O is C_3, hence there are $|O|/|C_3| = 8$ elements in its class.

The conclusion from Table 1 is that only the pure rotational groups C_n, D_n, T, O and I need be considered. The subgroup structure of these groups is shown in Table 2, in the form of subgroup chains, where all proper subgroups are ordered by strict inclusion. These chains begin with a maximal subgroup, they may or may not branch, and they end in a cyclic group. Only nonisomorphic chains are shown. Normal subgroups (with respect to the main group) are indicated. These are the subgroups that are unions of conjugacy classes. Commutator subgroups are encircled. See Appendix B for further discussion of the commutator subgroup.

Table 3 gives character tables for some of the cyclic groups. These are the basic "building blocks" for all other character tables, shown in Tables 4 and 5. In these tables the irreducible representations are denoted by the Mulliken symbols (A, $B-$one-dimensional, $E-$two dimensional and T denotes a three dimensional representation [10]). A dashed line separates faithful (below the line) from non-faithful characters.

Tables 4 and 5 also contain the following additional information (compare [22]):

(a) The center Z, which is the subgroup formed by all classes of single elements.

Table 1. Classification of point symmetry groups by order. ~ and [denote isomorphism; ×, direct product. For orders 48–120, the list does not include the nonisomorphic abelian groups

Order	Point symmetry group	Direct product structure	Mathematical symbols [17]
1	C_1		$\boldsymbol{Z}_1$
2	$C_2 \sim C_s \sim C_i$		$\boldsymbol{Z}_2 \sim \boldsymbol{S}_2$
3	C_3		$\boldsymbol{Z}_3 \sim \boldsymbol{A}_3$
4	a) $C_4 \sim S_4$		$\boldsymbol{Z}_4$
	b) [$D_2 = \{e, c_2, c_2', c_2''\}$	$C_2 \times C_2$	$\boldsymbol{V}$
	$C_{2h} = \{e, c_2, \sigma_h, i\}$		
	$C_{2v} = \{e, c_2, \sigma_v, \sigma_v'\}$		
5	C_5		$\boldsymbol{Z}_5$
6	a) $C_6 \sim S_6 \sim C_{3h}$	$C_3 \times C_2$	$\boldsymbol{Z}_6$
	b) [$D_3 = \{e, 2c_3, 3c_2\}$		$\boldsymbol{S}_3$
	$C_{3v} = \{e, 2c_3, 3\sigma_v\}$		
7	C_7		$\boldsymbol{Z}_7$
8	a) $C_8 \sim S_8$		
	b) C_{4h}	$C_4 \times C_2$	$\boldsymbol{Z}_8$
	c) D_{2h}	$D_2 \times C_2$	
	d) [$D_4 = \{e, 2c_4, c_2, 2c_2', 2c_2''\}$		$\boldsymbol{D}_4$
	$C_{4v} = \{e, 2c_4, c_2, 2\sigma_v, 2\sigma_d\}$		
	$D_{2d} = \{e, 2s_4, c_2, 2c_2', 2\sigma_d\}$		
10	a) $C_{10} \sim S_{10} \sim C_{5h}$	$C_5 \times C_2$	$\boldsymbol{Z}_{10}$
	b) [$D_5 = \{e, 2c_5, 2c_5^2, 5c_2\}$		$\boldsymbol{D}_5$
	$C_{5v} = \{e, 2c_5, 2c_5^2, 5\sigma_v\}$		
12	a) $C_{12} \sim S_{12}$	$C_4 \times C_3$	$\boldsymbol{Z}_{12}$
	b) C_{6h}	$C_6 \times C_2$	
	c) [$D_6 = \{e, 2c_6, 2c_3, c_2, 3c_2', 3c_2''\}$	$D_3 \times C_2$	$\boldsymbol{D}_6$
	$D_{3h} = \{e, 2s_3, 2c_3, \sigma_h, 3c_2, 3\sigma_v\}$		
	$C_{6v} = \{e, 2c_6, 2c_3, c_2, 3\sigma_v, 3\sigma_d\}$		
	$D_{3d} = \{e, 2s_6, 2c_3, i, 3c_2, 3\sigma_d\}$		
	d) $T = \{e, 4c_3, 4c_3^2, 3c_2\}$		$\boldsymbol{A}_4$
16	a) $C_{16} \sim S_{16}$		$\boldsymbol{Z}_{16}$
	b) C_{8h}	$C_8 \times C_2$	
	c) D_{4h}	$D_4 \times C_2$	
	d) [$D_8 = \{e, 2c_8, 2c_4, 2c_8^3, c_2, 4c_2', 4c_2''\}$		$\boldsymbol{D}_8$
	$C_{8v} = \{e, 2c_8, 2c_4, 2c_8^3, c_2, 4\sigma_v, 4\sigma_d\}$		
	$D_{4d} = \{e, 2s_8, 2c_4, 2s_8^3, c_2, 4c_2', 4\sigma_d\}$		
24	a) $C_{24} \sim S_{24}$	$C_8 \times C_3$	$\boldsymbol{Z}_{24}$
	b) C_{12h}	$C_{12} \times C_2$	
	c) D_{6h}	$D_6 \times C_2$	
	d) T_h	$T \times C_2$	
	e) $D_{12} \sim C_{12v} \sim D_{6d}$		$\boldsymbol{D}_{12}$
	f) [$O = \{e, 8c_3, 3c_2, 6c_4, 6c_2'\}$		$\boldsymbol{S}_4$
	$T_d = \{e, 8c_3, 3c_2, 6s_4, 6\sigma_d\}$		
48	O_h	$O \times C_2$	
60	$I = \{e, 12c_5, 12c_5^2, 20c_3, 15c_2\}$		$\boldsymbol{A}_5$
120	I_h	$I \times C_2$	

Table 2. Subgroup chains for nonabelian symmetry groups. Normality ($\rhd$) refers to the main group (not to the preceding subgroup). Commutator subgroups encircled

Dihedral	Cubic
$D_3 \rhd C_3$	$T \rhd D_2 \supset C_2$
$\supset C_2$	$\supset C_3$
$D_4 \rhd D_2 \rhd C_2$	$O \rhd T \rhd D_2 \supset C_2$
$\rhd C_4 \rhd C_2$	$\supset C_3$
	$\supset D_4 \supset C_4$
$D_8 \rhd D_4 \supset D_2 \rhd C_2$	$I \supset T \supset D_2 \supset C_2$
$\rhd C_4 \rhd C_2$	$\supset C_3$
$\rhd C_8 \rhd C_4 \rhd C_2$	$\supset D_5 \supset C_5$
	$\supset C_2$
	$\supset D_3 \supset C_3$
	$\supset C_2$

(b) All factor groups. Each is opposite the non-faithful irreducible representation(s) with the corresponding kernel. The factor group with respect to the commutator subgroup (encircled) gives all one-dimensional irreducibles. The factor group(s) with respect to the minimal normal subgroup(s) give all non-faithful irreducible characters. For further discussion of these points see Appendix B.

(c) The reduction of the subdued faithful (and some of the non-faithful) irreducible characters on the maximal subgroups $M \subset G$. (Cf. Table X-14 in [9].) The

(d) In Table 5, the classes are also denoted by the cycle structure. This helps in determining the fusion rules for classes of M into those of G. The correspondence is always of classes with the same cycle structure.

After this preliminary discussion we are ready to demonstrate the derivation of the irreducible characters for the dihedral and cubic groups. Since the usage of Eq. (3) for eliminating the non-irreducibles is clear, we always use the MMR instead. Our demonstration is therefore also a proof of its validity for all point symmetry groups.

Dihedral groups, D_n (order $2n$)

All dihedral groups have the same structure, depending on whether n is odd or even, and hence there are two types of character tables (Table 4).

1. Odd n. There are $(n+3)/2$ classes: c_n^k is in the same class with c_n^{-k} [there are $(n-1)/2$ such classes], nc_2 elements form a class and, of course, the identity. The maximal subgroups are C_n and C_2 (there are actually n conjugate C_2 subgroups). C_n is also the commutator subgroup.

Since $D_n/C_n \sim C_2$ ($\sim$ denotes isomorphism) there are two linear irreducible characters (Appendix B). This conclusion is easily reached in the composition method. The classes c_n^k and c_n^{-k} of C_n fuse into the same class in D_n. When n is odd, the trivial (principal) representation A is the only real representation, assigning the same character to both classes (Rule (i)), hence we must take $(A)_{C_n}$.

Table 3. Character tables for cyclic groups C_n. $\varepsilon \equiv \exp(2\pi i/n)$, $\varepsilon^* = \exp(-2\pi i/n)$

C_2	e	c_2
A	1	1
B	1	−1

C_3	e	c_3	c_3^2
A	1	1	1
E	1	ε	ε^*
	1	ε^*	ε

C_4	e	c_4	c_2	c_4^3
A	1	1	1	1
B	1	−1	1	−1
E	1	i	−1	$-i$
	1	$-i$	−1	i

C_5	e	c_5	c_5^2	c_5^3	c_5^4
A	1	1	1	1	1
E_1	1	ε	ε^2	ε^{2*}	ε^*
	1	ε^*	ε^{2*}	ε^2	ε
E_2	1	ε^2	ε^*	ε	ε^{2*}
	1	ε^{2*}	ε	ε^*	ε^2

C_8	e	c_8	c_4	c_8^3	c_2	c_8^5	c_4^3	c_8^7
A	1	1	1	1	1	1	1	1
B	1	−1	1	−1	1	−1	1	−1
E_2	1	i	−1	$-i$	1	i	−1	$-i$
	1	$-i$	−1	i	1	$-i$	−1	i
E_1	1	ε	i	$-\varepsilon^*$	−1	$-\varepsilon$	$-i$	ε^*
	1	ε^*	$-i$	$-\varepsilon$	−1	$-\varepsilon^*$	i	ε
E_3	1	$-\varepsilon^*$	$-i$	ε	−1	ε^*	i	$-\varepsilon$
	1	$-\varepsilon$	i	ε^*	−1	ε	$-i$	$-\varepsilon^*$

C_6	e	c_6	c_3	c_2	c_3^2	c_6^5
A	1	1	1	1	1	1
B	1	−1	1	−1	1	−1
E_2	1	$-\varepsilon^*$	$-\varepsilon$	1	$-\varepsilon^*$	$-\varepsilon$
	1	$-\varepsilon$	$-\varepsilon^*$	1	$-\varepsilon$	$-\varepsilon^*$
E_1	1	ε	$-\varepsilon^*$	−1	$-\varepsilon$	ε^*
	1	ε^*	$-\varepsilon$	−1	$-\varepsilon^*$	ε

For C_2 there are two possibilities, A or B, and the derivation of the linear characters is complete.

For a two dimensional representation we can only take the sum of two conjugate representations of C_n. This is due both to the class fusion restriction, Rule (i), and to Clifford's theorem (Rule (v)). For each of these we can chose $2A$, $2B$ or $A+B$ of C_2. The MMR ensures that only the last combination gives an irreducible character (which therefore assumes the value 0 on the class nc_2). Since there are $(n-1)/2$ such combinations, we have obtained all $(n+3)/2$ irreducible characters.

2. Even n. There are $n/2+3$ classes: The elements of C_n are again conjugated in $n/2-1$ pairs (c_n^k with c_n^{-k}) but now both $e=c_n^n$ and $c_2=c_n^{n/2}$ are self conjugate. The perpendicular c_2 rotations now form two classes: $(n/2)c_2'$ and $(n/2)c_2''$. The maximal subgroups are C_n and $D_{n/p}$, where p is a prime divisor of n. (There are actually p isomorphic but non-conjugate $D_{n/p}$'s.) When $n/2$ is odd, $D_n = D_{n/2}\times C_2$. When $n=2^k$ D_n is a p-group. The commutator subgroup is always $C_{n/2}$.

Since $D_n/C_{n/2}\sim D_2$, there are four linear irreducibles, easily obtained from the character table of D_2. We therefore discuss the composition method only for the problematic case $n=2^k$. In this case there are two isomorphic non-conjugate maximal subgroups $D'_{n/2}$ and $D''_{n/2}$, corresponding to the two classes $(n/2)c_2'$

Table 4. Character tables for dihedral groups. Notations explained in text

D_2	e	c_2	c_2'	c_2''	$Z=D_2$
A	1	1	1	1	$D_2/D_2 \sim C_1$
B_1	1	−1	1	−1	$D_2/C_2 \sim C_2$
B_2	1	1	−1	−1	$D_2/C_2 \sim C_2$
B_3	1	−1	−1	1	$D_2/C_2 \sim C_2$

D_3	e	$2c_3$	$3c_2$	$Z=C_1$
A_1	1	1	1	$D_3/D_3 \sim C_1$
A_2	1	1	−1	$D_3/C_3 \sim C_2$
E	2	−1	0	$(E)_{C_3}(A+B)_{C_2}$

D_4	e	$2c_4$	c_2	$2c_2'$	$2c_2''$	$Z=C_2$		
A_1	1	1	1	1	1	$(A)_{C_4}(A)_{D_2'}(A)_{D_2''}$	$D_4/D_4 \sim C_1$	$D_4/C_2 \sim D_2$
A_2	1	1	1	−1	−1	$(A)_{C_4}(B_2)_{D_2'}(B_2)_{D_2''}$	$D_4/C_4 \sim C_2$	
B_1	1	−1	1	1	−1	$(B)_{C_4}(A)_{D_2'}(B_2)_{D_2''}$	$D_4/D_2 \sim C_2$	
B_2	1	−1	1	−1	1	$(B)_{C_4}(B_2)_{D_2'}(A)_{D_2''}$	$D_4/D_2 \sim C_2$	
E	2	0	−2	0	0	$(E)_{C_4}(B_1+B_3)_{D_2}$		

D_5	e	$2c_5$	$2c_5^2$	$5c_2$	$Z=C_1$		
A_1	1	1	1	1	$(A)_{C_5}(A)_{C_2}$	$D_5/D_5 \sim C_1$	$D_5/C_5 \sim C_2$
A_2	1	1	1	−1	$(A)_{C_5}(B)_{C_2}$		
E_1	2	$2\cos\frac{2\pi}{5}$	$2\cos\frac{4\pi}{5}$	0	$(E_1)_{C_5}(A+B)_{C_2}$		
E_2	2	$2\cos\frac{4\pi}{5}$	$2\cos\frac{2\pi}{5}$	0	$(E_2)_{C_5}(A+B)_{C_2}$		

D_8	e	$2c_8$	$2c_4$	$2c_8^3$	c_2	$4c_2'$	$4c_2''$	$Z=C_2$			
A_1	1	1	1	1	1	1	1	$(A)_{C_8}(A_1)_{D_4'}(A_1)_{D_4''}$	$D_8/D_8 \sim C_1$	$D_8/C_4 \sim D_2$	$D_8/C_2 \sim D_4$
A_2	1	1	1	1	1	−1	−1	$(A)_{C_8}(A_2)_{D_4'}(A_2)_{D_4''}$	$D_8/C_8 \sim C_2$		
B_1	1	−1	1	−1	1	1	−1	$(B)_{C_8}(A_1)_{D_4'}(A_2)_{D_4''}$	$D_8/D_4 \sim C_2$		
B_2	1	−1	1	−1	1	−1	1	$(B)_{C_8}(A_2)_{D_4'}(A_1)_{D_4''}$	$D_8/D_4 \sim C_2$		
E_2	2	0	−2	0	2	0	0	$(E_2)_{C_8}(B_1+B_2)_{D_4}$			
E_1	2	$\sqrt{2}$	−2	$-\sqrt{2}$	−2	0	0	$(E_1)_{C_8}(E)_{D_4}$			
E_3	2	$-\sqrt{2}$	−2	$\sqrt{2}$	−2	0	0	$(E_3)_{C_8}(E)_{D_4}$			

and $(n/2)c_2''$ of D_n (e.g., the two classes $(n/2)c_2'$ and $(n/2)c_2''$ of $D'_{n/2}$ fuse into $(n/2)c_2'$ of D_n). From the character table of C_n we can now choose either A or B. For each of these we can take either A_1 or A_2 for either of $D'_{n/2}$ and $D''_{n/2}$ (A or B_2 if $n=4$). Together we have eight combinations of class functions [with $\phi(g)=\pm 1$], of which only four are generalized characters (and therefore irreducible).

As discussed in Sect. 2, this problem arises because the groups under consideration are elementary. It can be resolved if we note that a c_n operation would bring the symmetry axes of $D'_{n/2}$ to coincide with those of $D''_{n/2}$ [i.e. $c_n\{(n/2)c_2'\} = \{(n/2)c_2''\}$]. Therefore, if we take the symmetric character A for C_n, those for $D'_{n/2}$ and $D''_{n/2}$ should both be symmetric or both antisymmetric. If we chose the antisymmetric character B for C_n, those for $D'_{n/2}$ and $D''_{n/2}$ should be one symmetric and the other antisymmetric. We conclude that there are four linear irreducible characters corresponding to $(A)_{C_n}(A_1)_{D'_{n/2}}(A_1)_{D''_{n/2}}$, $(A)_{C_n}(A_2)_{D'_{n/2}}(A_2)_{D''_{n/2}}$, $(B)_{C_n}(A_1)_{D'_{n/2}}(A_2)_{D''_{n/2}}$ and $(B)_{C_n}(A_2)_{D'_{n/2}}(A_1)_{D''_{n/2}}$.

The remaining $n/2-1$ irreducible characters are 2-dimensional, derived (as for the case of odd n) from the complex conjugate pairs of irreducible representations of C_n. Now all isomorphic $D_{n/2}$ have the same decomposition. It would be either A_1+A_2 (for odd $n/2$), B_1+B_2 (for even $n/2$) or one of the 2-dimensional irreducible representations, depending on the value of ϕ for the c_n elements. In all cases we obtain $\phi(c'_2)=\phi(c''_2)=0$. We conclude that all the class functions obtained are irreducible characters. The character tables for the dihedral groups are therefore complete.

Cubic groups (Table 5)

1. The tetrahedral group $T\sim A_4$ (order 12). The two maximal subgroups are C_3 (there are 4 conjugate C_3's) and D_2. Each class of C_3 fuses into a different class of T. All three c_2 classes of D_2 fuse into the single class $3c_2$ of T. Let us start with the one-dimensional class functions.

Table 5. Character tables for the cubic groups. Notations explained in text

T	(1^4) e	$(1,3)$ $4c_3$	$(1,3)$ $4c_3^2$	(2^2) $3c_2$	$Z=C_1$	
A	1	1	1	1	$(A)_{C_3}(A)_{D_2}$	$T/D_2\sim C_3$
E	1	ε	ε^*	1	$(E)_{C_3}(2A)_{D_2}$	
	1	ε^*	ε	1		
T	3	0	0	-1	$(E+A)_{C_3}(B_1+B_2+B_3)_{D_2}$	

O	(1^4) e	$(1,3)$ $8c_3$	(2^2) $3c_2$	(4) $6c_4$	$(1^2,2)$ $6c'_2$	$Z=C_1$		
A_1	1	1	1	1	1	$(A)_T(A_1)_{D_4}$	$O/T\sim C_2$	$O/D_2\sim C_3$
A_2	1	1	1	-1	-1	$(A)_T(B_1)_{D_4}$		
E	2	-1	2	0	0	$(E)_T(A_1+B_1)_{D_4}$		
T_1	3	0	-1	1	-1	$(T)_T(E+A_2)_{D_4}$		
T_2	3	0	-1	-1	1	$(T)_T(E+B_2)_{D_4}$		

I	(1^5) e	(5) $12c_5$	(5) $12c_5^2$	$(1^2,3)$ $20c_3$	$(1,2^2)$ $15c_2$	$Z=C_1$
A	1	1	1	1	1	$I/I\sim C_1$
T_1	3	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	0	-1	$(T)_T(E_1+A_2)_{D_5}$
T_2	3	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	0	-1	$(T)_T(E_2+A_2)_{D_5}$
G	4	-1	-1	1	0	$(T+A)_T(E_1+E_2)_{D_5}$
H	5	0	0	-1	1	$(T+E)_T(E_1+E_2+A_1)_{D_5}$

$\nu=1$. The only linear character of D_2 assigning the same value to these three classes is A (rule (i)), which may be combined with any linear character of C_3. Hence T has three linear characters.

$\nu=2$. There can be no 2-dimensional $\chi \in \mathrm{Irr}(T)$, since no combination, except $2A$, of two irreducible characters of D_2 assigns the same value to the three c_2 elements. The combination $(2A)_{D_2}$ can be discarded for three different reasons:

(a) We know that D_2 is both the commutator subgroup and the only minimal normal subgroup of T. Hence all non-linear irreducible characters must be faithful.

(b) If this combination would give an irreducible character, we would complete the character table of T without using the other irreducible characters of D_2, in contradiction to Rule (iii).

(c) Complete the class function by taking a combination of any two linear characters of C_3. But since they were already used for the linear characters of T, the result is reducible.

$\nu=3$. The 3-dimensional class functions that need be considered are composed of $(B_1+B_2+B_3)_{D_2}$. (The combination $(3A)_{D_2}$ contradicts the MMR and leads, as above, to a reducible character.) For C_3 we can take any combination of three irreducibles, and the MMR now ensures that the only function which would be an irreducible character is $(B_1+B_2+B_3)_{D_2}(A+E)_{C_3}$, which contains each linear character of C_3 once. Alternatively, we could have rejected $(3A)_{C_3}$ as being non-faithful, and any combination of an odd number of the non-real linear characters of C_3 as leading to an irreducible character which is not real. This is in contrast to it being the last irreducible representation (hence its complex conjugate must equal to itself in order to belong to $\mathrm{Irr}(T)$).

We conclude that there is one 3-dimensional irreducible representation and the character table for T is complete.

2. The octahedral group $\mathrm{O} \sim \mathrm{S}_4$ (order 24). The maximal subgroups are T and D_4 (there are actually 3 conjugate D_4's). The two classes $4c_3$ and $4c_3^2$ of T fuse into the class $8c_3$ of O. The classes c_2 and $2c_2'$ of D_4 fuse into the class $3c_2$ of O, while the class $2c_2''$ of D_4 is partial to $6c_2'$ in O.

$\nu=1$. The only possible combination (Rule (i)) from T is $(A)_T$. This gives $\chi(c_2)=1$, with the result (Rules (i) and (ii)) of two possible combinations from D_4: A_1 and B_1. Hence O has two linear characters.

$\nu=2$. Possible combinations (Rule (i)) from T are $(2A)_T$ or $(E)_T$. In both cases $\phi(c_2)=2$, with the result (Rules (i) and (ii)) of only one combination from D_4: A_1+B_1. The class function $(2A)_T(A_1+B_1)_{D_4}$ is now immediately rejected by the MMR, or because it is the sum of the two linear characters obtained in the previous step (Rule (iv)).

$\nu=3$. The class functions that agree with Rule (i) may have either $(A+E)_T$ or $(T)_T$. We argue that $(A+E)_T$ would not correspond to an irreducible character for several reasons:

(a) The intersection of the kernels of all three irreducibles obtained in the previous stages is exactly D_2, which is a minimal normal subgroup. Hence all additional $\chi \in \mathrm{Irr}\,(O)$ must be faithful.

(b) $T \lhd O$, hence this contradicts Clifford's theorem (Rule (v)).

(c) Any combination from D_4 with $\nu=3$ would give $\phi(c_4)\neq 0$. Since we have obtained a linear character (A_2) for O which assigns -1 to the element c_4, we conclude that if there is a 3-dimensional irreducible character, there will be two such characters (one obtained by multiplying the other by A_2). But then we would exhaust all $\chi \in \mathrm{Irr}\,(O)$ without using $(T)_T$, in contradiction to Rule (iii).

(d) Finally, $(A+E)_T$ is also rejected by the MMR.

We are left with $(T)_T$, assigning $\phi(c_2)=-1$. Therefore, by Rules (i) and (ii) we can have only $(E+A_2)_{D_4}$ or $(E+B_2)_{D_4}$, which completes the character table for the octahedral group.

3. The icosahedral group $I \sim A_5$ (order 120). The classes of maximal subgroups are T, D_5 and D_3. From Table 2 we see that each subgroup of D_3 is also a subgroup of T. Hence the ME-cover for I does not include D_3.

$\nu=1$. A_5 is simple and hence has only one linear character, A. This can also be deduced by the composition method, since the two classes $4c_3$ and $4c_3^2$ of T fuse into the single class $20c_3$ of I.

$\nu=2$. I has no irreducible character of this dimension, since T has no faithful character with $\nu=2$.

$\nu=3$. For the same reason we can have only $(T)_T$. Note that the possible (by Rules (i) and (ii)) class functions $(A+E)_T(3A_1)_{D_5}$ and $(3A)_T(3A_1)_{D_5}$ violate the MMR for both maximal subgroups. Also $(3A)_T(3A_1)_{D_5}$ is clearly reducible. $(T)_T$ implies $\phi(c_2)=-1$, so we can choose from D_5 only E_1+A_2, E_2+A_2 or A_1+2A_2. The last combination is discarded on the basis of the MMR (or because it is non-faithful). We are left with two 3-dimensional irreducible characters of I.

$\nu=4$. The only faithful combination from T is $(T+A)_T$. Note, again, that a class function such as $(2E)_T(4A_1)_{D_5}$ would be rejected also by the MMR. Now, $(T+A)_T$ implies $\phi(c_2)=0$. Of all possible combinations from D_5, the MMR excludes all but $(E_1+E_2)_{D_5}$. (Some of the other combinations could again be discarded for other reasons: Being non-faithful, $\mathbb{N}$-linear combinations of other irreducibles, etc.). We are therefore left with only one 4-dimensional irreducible character.

$\nu=5$. This will be the last irreducible character for I. The only combination from T to give an irreducible character is $(T+E)_T$: We have not used $(E)_T$ yet (Rule (iii)) and $(2E+A)_T$ is rejected both by the MMR and for its non-faithfulness. $(T+E)_T$ gives $\phi(c_2)=1$. Of the combinations that can be chosen (Rule (ii)) from D_5, only $(E_1+E_2+A_1)_{D_5}$ agrees with the MMR. The character table of the icosahedral group is now complete.

5. Double groups

Obtaining the characters for the two-valued representations of Bethe's double groups (see Sect. 9-7 in [4]) is particularly easy in the present method. The reason is that these characters are orthogonal to those of the single-valued representations (obtained in Sect. 4), and that a subdued two-valued character decomposes into two-valued characters of the subgroup. Hence one only needs to consider the two-valued irreducible representations of the maximal subgroups.

In the present method, one first constructs the character tables for the cyclic double-groups C'_n. This is easily done since $C'_n \sim C_{2n}$. All their double-valued representations are therefore one-dimensional. Next, one uses these to construct the characters of the two-valued irreducible representations of the double-dihedral and double-cubic groups.

To demonstrate this procedure, the reader is referred to Table 9-11 of [4] (the characters of c_4 and c_4^3 in the character table for D'_4 should be $\pm\sqrt{2}$).

(a) D'_3. The maximal subgroups of D'_3 are C'_3 and C'_2. The first has 3 one-dimensional two-valued irreducible representations, denoted by A' and E'. The second maximal subgroup, C'_2, has two such complex-conjugate representations, denoted jointly by E'. From these one can compose $(E')_{C'_2}(2A')_{C'_3}$ and $(E')_{C'_2}(E')_{C'_3}$, which are the two 2-dimensional irreducible representations of D'_3 denoted by E'_1 and E'_2, respectively. Note that $(E')_{C'_2}(2A')_{C'_3}$ is not rejected by the MMR, since it is actually a shorthand notation for two linear characters.

(b) T'. The maximal subgroups are C'_3 and D'_2. The latter has only one two-valued representation, E'. It is easily seen that T' can have three different 2-dimensional irreducible characters, obtained by composing $(E')_{D'_2}$ with the sum of any pair out of the three two-valued irreducible characters of C'_3. Thus $(E')_{T'}$ and $(G')_{T'}$ are obtained.

(c) O'. The maximal subgroups are T' and D'_4. Their two-valued irreducible characters can be composed as $(E')_{T'}(E'_1)_{D'_4}$, $(E')_{T'}(E'_2)_{D'_4}$ and $(G')_{T'}(E'_1+E'_2)_{D'_4}$, to yield the two-valued irreducible representations of O', denoted by E'_1, E'_2 and G', respectively. The combination $(2E')_{T'}(E'_1+E'_2)_{D'_4}$ is rejected by the MMR.

6. Conclusion

We have introduced the new "Ascending Symmetry" or "Composition" method for determining irreducible characters of (nonabelian) finite groups. It is a consequence of Brauer's theorem on generalized characters that the method is guaranteed to work for all non-elementary groups. Whether this is so when the group is (nonabelian but) elementary is not completely clear. We have also introduced a characterization of irreducible characters by the "Maximum Mixing Rule", which has an interesting physical interpretation as ensuring maximal entropy increase by external perturbations. The rule has therefore been checked for all the point symmetry groups of mathematical physics, whose character tables we have constructed by this new method, as well as for the crystal double-groups. It may be interesting to know to what extent it holds for a general finite group. Finally, it is unclear whether the composition method could have a practical application in computational group theory [16], for calculating character tables of large finite groups: For medium-size groups other methods may be more efficient, while for large groups the maximal sub-groups and their fusion rules are unknown.

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Appendix A: Definitions of group theoretic notions

—Ambivalent class: conjugacy class which equals to its inverse.
—Centralizer (of an element $g \in G$): all $x \in G$ that commute with g.
—Class function: a function assigning a complex number to each conjugacy class.
—Commutator subgroup: generated by all elements of the form $xyx^{-1}y^{-1}$, where $x, y \in G$.
—Conjugate characters: if $H \lhd G$, ψ is a character of H, $h \in H$ and $g \in G$, then $\psi^g(h) = \psi(ghg^{-1})$ is a character conjugate to ψ in G.
—Coset: a right (left) coset of subgroup H in G is the set $Hg(gH)$, $g \in G$.
—Elementary subgroup: a direct product of a p-group and a cyclic group.
—Factor group: if $H \lhd G$ the factor group G/H is the set of all left (or right) cosets of H in G with multiplication by representatives: $(xH)(gH) = xgH$.
—Faithful representation: an isomorphism of G onto a group of matrices. Its kernel is the identity.
—Fusion of classes: unison of all classes of a subgroup H that are contained in a class of G.
—Generalized character: difference of characters. It is a $\mathbb{Z}$-linear combination of irreducible characters (i.e., the coefficients are integers, possibly negative).
—Kernel of a character χ: all $g \in G$ so that $\chi(g) = \chi(e)$.
—Maximal subgroup $H \subseteq G$: there is no subgroup $K \subset G$ so that $H \subset K$.
—Normal subgroup: $H \lhd G$ if it is self-conjugate, i.e. $H = gHg^{-1}$ for all $g \in G$.
—p-group: all its elements are of orders which are powers of the prime p.
—Proper subgroup: a subgroup $H \subseteq G$ is "proper" if it differs from G or $\{e\}$.
—Simple group: has no normal subgroups.
—Simply reducible: G is simply reducible on a subgroup H if all $\psi \in \mathrm{Irr}(H)$ appear at most once in χ_H, $\chi \in \mathrm{Irr}(G)$.
—Subclass: the subclass of $g \in G$ with respect to a subgroup $H \subseteq G$ are all elements hgh^{-1}, $h \in H$.
—Subduced character: a character χ_H of a subgroup $H \subseteq G$ is subduced from G if it is a restriction of a character χ of G to the classes of H.

Appendix B: Non-faithful characters and factor groups

We briefly review the procedure of obtaining the non-faithful irreducible characters from those of factor groups. This is based on the homomorphism theorem [17]: On the one hand the kernel of every homomorphism (i.e., the set of elements mapped to the identity) is a normal subgroup. On the other hand, every $H \lhd G$ defines a "canonical" homomorphism from G to the factor group G/H whose kernel is H, by $g \to gH$. Every homomorph of G with kernel H is thus isomorphic to G/H. Since a representation is a homomorphism (onto a group of matrices), and homomorphism is transitive, one concludes that if $\chi \in \mathrm{Irr}(G/H)$ then $\chi(gH) \in \mathrm{Irr}(G)$. For $H \neq \{e\}_1$ χ is non-faithful.

To obtain all non-faithful irreducible representations one needs to consider only the minimal normal subgroups. This follows from the theorem stating [17] that if K and H are both normal subgroups of G, $K \subset H \subset G$, then $H/K \lhd G/K$ and

$$G/H \sim (G/K)/(H/K) \tag{B1}$$

This implies that all irreducible representations of G/H are (nonfaithful) irreducible representations of G/K.

Finally, all linear irreducible characters can be found from the factor group with respect to the commutator subgroup H_c, which is [17] the (unique) normal subgroup containing all elements $a_1 a_2 \cdots a_n a_1^{-1} a_2^{-1} \cdots a_n^{-1}$ for all $a_i \in G$, $n \geq 2$. Alternatively, if $\mathscr{C}_i$ is a class in G and $\mathscr{C}_{i'}$ the class of inverses, H_c is generated by all $\mathscr{C}_i \mathscr{C}_{i'}$. This follows from the theorem [17] that if $H \lhd G$ then G/H is abelian (hence all its characters are linear) if and only if $H \supset H_c$. (Also $H \supset H_c \Rightarrow H \lhd G$). In particular, the number of one-dimensional irreducible representations equals $|G|/|H_c|$.

As an example consider D_8 and $C_4 \lhd D_8$. C_4 is the commutator subgroup of D_8. To see this, take all elements of the form $\mathscr{C}_i \mathscr{C}_{i'}$ (note that D_8 is ambivalent, $\mathscr{C}_i = \mathscr{C}_{i'}$): $\{2c_4\}^2 = \{e, c_2\}$, $\{2c_8\}^2 = \{e, 2c_4\}$, etc. Going over all classes of D_8 one obtains exactly all elements of C_4.

Consider next the factor group D_8/C_4. It contains 4 distinct cosets: $eC_4 = \{e, 2c_4, c_2\}$, $c_8 C_4 = \{2c_8, 2c_8^3\}$, $c_2 C_4 = \{4c_2'\}$ and $c_2'' C_4 = \{4c_2''\}$. (Since D_8/C_4 is abelian, each coset is made of whole classes of D_8.) D_8/C_4 must be isomorphic to C_4 or D_2. eC_4 is its identity element. Pick a representative from the other three cosets. Their square belongs to eC_4, hence $D_8/C_4 \sim D_2$.

We construct the linear characters of D_8 from the character table of D_2 as follows: Take the column of the identity in D_2 and put it under the classes $e, 2c_4$ and c_2 of D_8; take the column of the class c_2' in D_2 and use it for the classes $2c_8$ and $2c_8^3$ of D_8; finally, put the columns of c_2 and c_2'' in D_2 under $4c_2'$ and $4c_2''$ of D_8, respectively.

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